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Abstract

In this article we present the Bayesian decision theoretic setup for design of accelerated life tests. We review some of the key contributions to the Bayesian design of accelerated life tests. In so doing, we discuss approximate Bayesian designs based on linear Bayesian methods and Monte Carlo based methods. We consider computational issues regarding the evaluation of expectation and optimization steps in the solution of the decision problem and discuss some recent Monte Carlo approaches that can reduce the computational effort in the design problem.

Key Words: Loss funtion, linear Bayes, preposterior analysis, sequential design, Markov chain Monte Carlo.

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[1]Contribution to *The Encyclopedia of Statistics in Quality and Reliability* (Eds. F. Ruggeri, R. Kenett and F. Faltin), John Wiley & Sons, UK.

1. Introduction and Overview

Accelerated life tests (ALT's) involve testing systems in an environment that is more severe than the use environment and using the data collected in the accelerated environment for inferring failure behavior in the use environment. The design problem in accelerated life testing is concerned with specification of the number and magnitude of the accelerated stress levels, and the number of items to be tested at these stress levels.

Most of the Bayesian literature in ALT's have focused on developing inference methods; see for example Mazzuchi and Soyer (1992), van Dorp et al. (1996), Mazzuchi, Soyer and Vopatek (1997), and van Dorp and Mazuuchi (2004a, 2004b). The majority of the work published on design of ALT's, relied on sample theoretic methods; see for example the text by Nelson (1990) and the recent bibliography of accelerated testing plans by Nelson (2005a, 2005b). Exceptions to these are the earlier Bayesian papers by Martz and Waterman (1978), DeGroot and Goel (1979) and more recent Bayesian approaches by Menzefricke (1991), Chaloner and Larntz (1992), Polson (1993), Verdinelli, Polson and Singpurwalla (1993), Soyer and Vopatek (1995), Erkanli and Soyer (2000) and Zhang and Meeker (2006).

Most of these Bayesian approaches are based on the theory of optimal Bayesian designs for linear models as in Chaloner (1984). Thus, the results are applicable to ALT designs when the life model is normal or lognormal. For example, Chaloner and Larntz (1990) consider Bayesian designs for Type I censored tests when there is uncertainty about whether the underlying life model is lognormal or Weibull and several fractiles of the lifelength distribution at the use stress are of interest. The optimality criterion considered by authors is proportional to the expected asymptotic variance of the fractiles of interest. Designs satisfying this criterion are identified using numerical methods. This approach can be considered as a Bayesian version of the sample theoretic methods considered in Nelson (1990). Menzefricke (1991) formulates a Bayesian approach to the optimal design of Type II censored ALTs for use when the lifelength model is lognormal. Verdinelli, Polson, Singpurwalla (1993) identify a design for a complete ALT that maximizes utility, as represented by Shannon information, given the usual linear model assumptions. More recent work by Zhang and Meeker (2006) also considers some large sample results for Bayesian ALT designs as well as simulation based methods.

As noted by Vopatek (1992) and Soyer and Vopatek (1995), if the underlying life models are exponential or Weibull, nonlinearities arise in the analysis, and then optimal designs can be obtained by use of either numerical methods or by special techniques such as linear Bayesian methods. Recent advances in statistical computing, nonparametric surface estimation and implementation of Markov chain Monte Carlo methods

contributed to development of computationally efficient methods for optimal designs. For example, recent simulation based approach of Muller and Parmigiani (1995) and its extensions presented by Muller (1998) were considered in ALT designs. Erkanli and Soyer (2000) used this approach for fixed (nonsequential) designs and developed an extension to sequential designs.

In this article we will review some recent Bayesian approaches in ALT designs. In so doing, we present the Bayesian decision theoretic setup for the optimal ALT design problem following Polson (1993) and Erkanli and Soyer (2000). We adopt this setup to a single point design problem and illustrate difficulties involved in evaluation of preposterior losses in implementation of the Bayesian paradigm. We present linear Bayesian methods and Monte Carlo based approaches to alleviate some of these difficulties. Finally we illustrate the implementation of a simulation-based design algorithm using the approach of Muller and Parmigiani (1995).

2. Bayesian Decision Theoretic Setup for the Optimal Design Problem

In Bayesian paradigm, the optimal design problem can be viewed as a decision problem in the sense of Lindley (1985). Thus, the optimal designs are chosen by maximizing expected utility. As noted by Polson (1993), this provides a formal approach to the design of experiments. A comprehensive review of Bayesian experimental design can be found in Chaloner and Verdinelli (1995).

Erkanli and Soyer (2000) point out that the Bayesian approach to the optimal design problem requires specification of three components:

(i) a *utility (loss) function* that reflects the consequences of selecting a specific design;
(ii) a probability model;
(iii) a prior distribution reflecting designer's *a priori* beliefs about all unknown quantities.

Let d denote a specific design, that is, the decision variable, λ_u, the failure characteristic of interest (such as the use environment failure rate) and, a an action based on data (for example, a prediction for λ_u). We denote the utility function as $U(\lambda_u, d, a)$ and the probability model as $p(D|\lambda_u, d)$ where D is the data observed from experiment d. Prior beliefs about λ_u is described by the prior distribution $p(\lambda_u)$. It is possible that λ_u may be a vector or a function of several parameters. Once the design d is specified and the ALT is performed, data D is observed and uncertainty about λ_u is revised according

to the laws of probability. Then, the optimal action a is chosen based on the data. This process can be depicted by the decision tree in Figure 1.

In Figure 1, the decision node D_2 represents selection of the Bayes rule a given data. For the case of squared error loss $L(\lambda_u, d, a)$, the utility function is $U(\lambda_u, d, a) = -(\lambda_u - a)^2$ and the optimal Bayes rule a^* is the posterior mean. Then the last two nodes can be replaced by $U(\lambda_u, d, a^*) = -V(\lambda_u | D, d)$ which is the posterior variance of λ_u and the optimal design is chosen by minimizing the preposterior variance

$$\mathrm{E}_{D|d}[V(\lambda_u | D, d)] = \int V(\lambda_u | D, d)\, p(D|\lambda_u, d)\, p(\lambda_u)\, \mathrm{d}\lambda_u\, \mathrm{d}D. \qquad (1)$$

In other words, the optimal design is given by $d^* = \text{argmin}\{\mathrm{E}_{D|d}[V(\lambda_u | D, d)]\}$.

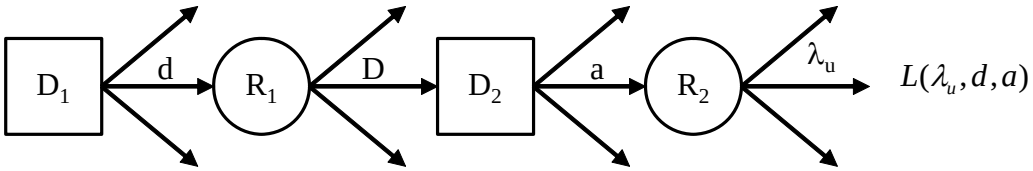

Figure 1: Decision Tree for the Design Problem.

As pointed out by Erkanli and Soyer (2000), this solution is obtained as a result of "folding back" the decision tree through taking expectations at random nodes and maximizing the expected utility at the decision nodes of Figure 1. The above setup can be adopted for any form of the utility function $U(\lambda_u, d, a)$. For example, Verdinelli, Polson and Singpurwalla (1993) used the Shannon's information as the utility functiom. Thus, in general the optimal design is obtained as

$$d^* = \underset{d}{\text{argmax}} \left\{ \mathrm{E}_{D|d} \left[U(\lambda_u, d, a^*) | D, d \right] \right\}, \qquad (2)$$

where $a^* = \text{argmax} \left\{ \mathrm{E}_{\lambda_u | D, d} [U(\lambda_u, d, a) | D, d] \right\}$. In what follows we will present an example of the decision theoretic setup assuming exponential life times.

2.1 Example: Single Stress ALT Designs

An important component of the ALTs is the *time transformation function (or the acceleration function)* that describes the relationship between the failure characteristic of interest and the applied level of the stress variable.

Let λ_i denote the failure rate at the accelerated stress environment S_i, assume that life length X_i is exponential with λ_i and the time transformation function is given by the power law model

$$\lambda_i = \alpha S_i^{\beta} . \tag{3}$$

Let us consider the design of an ALT where n items will be tested until failure at a single accelerated stress environment S_i with the purpose of making inference about $\lambda_u = \alpha S_u^{\beta}$, the failure rate at the use stress environment. Such single point designs have been considered by Martz and Waterman (1978).

In the above case the design problem is to find the stress level S_i minimizing the preposterior variance of λ_u. In other words, in our setup, we assume that the design variable $d = S_i$ and the loss function is given by $L(\lambda_u, d, a) = (\lambda_u - a)^2$. Thus, the design problem requires computation of

$$S_i^* = d^* = \text{argmin}\{\text{E}_{D|d}[\, V(\lambda_u|\, D,\, d)]\}. \tag{4}$$

In other words, our decision problem in Figure 1 reduces to a single stage problem. For illustrative purposes, we assume that β is known and we specify a gamma prior distribution for α with shape parameter a and scale b. Given n failures $D = (x_1, x_2, \ldots, x_n)$ the posterior distribution of α is given by a gamma density denoted as $Gamma[(a+n),\ (b + S_i^{\beta}\, T_i)]$, where T_i is the total time on test at stress environment S_i. Using the power law model, the posterior variance of λ_u is

$$V(\lambda_u|\, D,\, S_i) = V(\alpha S_u^{\beta}|\, D,\, S_i) = \frac{(a+n)\,(S_u/S_i)^{2\beta}}{(T_i + \frac{b}{S_i^{\beta}})^2} . \tag{5}$$

It can be shown that the preposterior variance $\text{E}_{D|d}[V(\lambda_u|\, D,\, S_i)]$ is given by

$$V(\lambda_u|\, S_i) = \frac{a(a+1)\, S_u^{2\beta}}{(a+n+1)\, b^2} , \tag{6}$$

where the expectation is with respect to the predictive distribution of T_i given S_i [Erkanli and Soyer (2000)]. Since (6) is not a function of the stress S_i, it does not matter at what stress level the items are tested. This is due to the fact that β is assumed to be known in the power law model. This result was also noted by Vopatek (1992) and Verdinelli, Polson and Singpurwalla (1993) for the special case of the power law with $\beta = 1$.

In general, except in few special cases, the optimal designs in (1) and (2) can not be obtained analytically. In ALT models where the underlying failure distribution is exponential or Weibull, posterior and/or predictive inferences are not in closed form.

Thus, evaluation of the designs in (1) and (2) requires either use of approximation techniques such as linear Bayesian methods as in Soyer and Vopatek (1995) or Monte Carlo methods as in Erkanli and Soyer (2000) and Zhang and Meeker (2006).

3. Linear Bayesian Designs for ALTs

In the power law model example of Section 2.1, if the coefficient β is unknown then the posterior variance $V(\lambda_u|\, D,\, S_i)$ can not be obtained in closed form for any prior specification for β in the case of exponential lifetimes. One way to deal with this problem is to consider approximate Bayesian designs using linear Bayesian methods.

Soyer and Vopatek (1995) considered the power law model (3) with exponential life times where both α and β are treated as unknown. The authors described their uncertainty about these unknown quantities partially by specifying only the first and second-order moments. More specifically, they consider the log transformation of the power law model as

$$\eta_i = log\lambda_i = log\alpha + \beta logS_i = \boldsymbol{F}_i'\boldsymbol{\theta}, \tag{7}$$

where $\boldsymbol{F}_i' = (1\ logS_i)$ and $\boldsymbol{\theta} = (log\alpha\ \beta)$. Prior to testing at S_i distribution of $\boldsymbol{\theta}$ is partially specified as $\boldsymbol{\theta} \sim (\boldsymbol{m}_0,\, \boldsymbol{C}_0)$. Following Mazzuchi and Soyer (1992), the prior moments can be used in (7) to specify a loggamma distribution as the prior for η_i with parameters a_i and b_i selected such that

$$\Psi(a_i) - logb_i = \boldsymbol{F}_i'\boldsymbol{m}_0 \text{ and } \Psi'(a_i) = \boldsymbol{F}_i'\boldsymbol{C}_0\boldsymbol{F}_i, \tag{8}$$

where $\Psi(\bullet)$ and $\Psi'(\bullet)$ are the digamma and trigamma functions, respectively. Note that complete specification of the prior for η_i (and thus for λ_i) enables us to obtain the posterior distribution for η_i after testing at S_i. Under the assumption of no censoring, standard Bayesian conjugate analysis shows that the posterior distribution of η_i is a loggamma density with parameters $(a_i + n)$ and $(b_i + T_i)$, where n is the total number of items tested under environment S_i and T_i is the total time on test.

In order to obatin the posterior variance $V(\eta_u|\, D,\, S_i)$ or $V(\lambda_u|\, D,\, S_i)$, it is necessary to update $\boldsymbol{\theta} = (log\alpha\ \beta)$. Since prior distribution of $\boldsymbol{\theta}$ is only partially specified through moments, one can only update the moments given the data. This updating is done via using the linear Bayesian methods. Soyer and Vopatek (2005) show that the posterior moments of $\boldsymbol{\theta}$ are given by

$$\boldsymbol{m}_i = \boldsymbol{m}_0 + \boldsymbol{h}_i \frac{E(\eta_i|D) - E(\eta_i)}{V(\eta_i|D)} \tag{9}$$

$$\boldsymbol{C}_i = \boldsymbol{C}_0 - \boldsymbol{h}_i \boldsymbol{h}_i' \, \frac{1 - V(\eta_i|D)/V(\eta_i)}{V(\eta_i|D)}, \tag{10}$$

where $\boldsymbol{h}_i = \boldsymbol{C}_0 \boldsymbol{F}_i$. Thus, the posterior distribution of $\boldsymbol{\theta}$ is partially specified by the two moments as $(\boldsymbol{\theta}|D) \sim (\boldsymbol{m}_i,\, \boldsymbol{C}_i)$.

Once such posterior specification is available, we can use the fact that at the use stress we have $\eta_u = \boldsymbol{F}_u' \boldsymbol{\theta}$, where $\boldsymbol{F}_u' = (1 \;\; log S_u)$ and obtain the posterior variance of η_u. We can show that $V(\eta_u|\, D,\, S_i)$ is given by

$$\boldsymbol{F}_u' \boldsymbol{C}_0 \boldsymbol{F}_u - (\boldsymbol{F}_u' \boldsymbol{C}_0 \boldsymbol{F}_i)^2 \Big[\frac{1 - \Psi'(a_i + n)/\boldsymbol{F}_i' \boldsymbol{C}_0 \boldsymbol{F}_i}{\boldsymbol{F}_i' \boldsymbol{C}_0 \boldsymbol{F}_i} \Big] \tag{11}$$

which implies that as n increases the posterior variance decreases for any level of S_i as expected. We note that the posterior variance is not a function of the prior mean $\boldsymbol{m}_0$ but it depends on prior covariance matrix $\boldsymbol{C}_0$. Furthermore, the above is not a function of the total time on test T_i. As a result of this the posterior variance is the same as the preposterior variance, that is, $V(\eta_u|\, D,\, S_i) = V(\eta_u|\, S_i)$. In other words, one can find the single point optimal design by simply minimizing (11) with respect to S_i.

Different forms of the prior covariance matrix $\boldsymbol{C}_0$ were considered by Soyer and Vopatek (2005) and the corresponding optimal designs were presented. For example in the special case of the power law model where α is known, say $\alpha = 1$, it can be shown that the optimal design is to test all the items the highest possible stress level S_H, that is, $S_i^* = S_H$. If $log\alpha$ and β are assummed to be independent apriori, that is, if $\boldsymbol{C}_0$ is a diagonal matrix, then it was shown by Vopatek (1992) that for large n an optimal design can be obtained as

$$S_i^* = S_u^{\left[1 + 1/[nV(log\alpha)]\right]}. \tag{12}$$

The above implies that as the number of items to be tested is large then the optimal stress is closer to the use stress S_u. Similarly, as prior uncertainty about $log\alpha$ increases, the optimal stress level moves closer to S_u.

If the loss function is specified in terms of the posterior variance of λ_u rather than of η_u then it can be shown that $V(\lambda_u|\, D,\, S_i)$ is a function of the total time on test. More specifically in order to obtain $V(\lambda_u|\, D,\, S_i)$, it is necessary to have a distributional form for η_u and thus for λ_u. This is done after updating by specifying a loggamma distribution for η_u with parameters a_u and b_u selected such that

$$\Psi(a_u) - log b_u = \boldsymbol{F}_u' \boldsymbol{m}_i \text{ and } \Psi'(a_u) = \boldsymbol{F}_u' \boldsymbol{C}_i \boldsymbol{F}_u. \quad (13)$$

The above implies a gamma posterior for λ_u implying that $V(\lambda_u | D, S_i) = a_u/(b_u)^2$. Using this form Vopatek (1992) obtained the preposterior variance $V(\lambda_u | S_i)$.

The linear Bayesian setup can be generalized to multiple point ALT designs where one considers m different stress levels where $n_1, n_2, \ldots, n_m$ are tested at stress environments $S_1, S_2, \ldots, S_m$. This can be done either in an adaptive manner by solving $m-$ sequential one point design problems or solving a fixed design problem. Some of these extensions as well as design of censored tests are considered in Vopatek (1992) using linear Bayesian methods.

4. Simulation Methods for Bayesian Designs

In the power law model of Section 2.1, if β is unknown then the preposterior variance of λ_u can not be obtained in closed form. One way to compute the posterior variance is to use a Markov Chain Monte Carlo (MCMC) method such as the Gibbs sampler.

The Gibbs sampler requires the full conditional distributions $p(\alpha \mid \beta, D)$ and $p(\beta | \alpha, D)$. Using a gamma prior for α as before the full conditional of α is given by the gamma distribution $Gamma[(a+n), (b + S_i^{\beta} T_i)]$. On the other hand, any prior distribution on β does not yield a standard form for the full conditional $p(\beta | \alpha, D)$, which is given by

$$p(\beta | \alpha, D) \propto S_i^{\beta n} \, exp(-\alpha S_i^{\beta} T_i) \, p(\beta). \quad (14)$$

Thus, a method such as the rejection sampling needs to be used to draw samples from $p(\beta | \alpha, D)$. In this particular case, the standard rejection sampling can be easily implemented since the maximum of the conditional likelihood $\mathcal{L}(\beta; \alpha, D)$ is analytically available. We can show that maximum of the conditional likelihood is given by

$$\widehat{\beta} = \frac{log(n) - log(\alpha T_i)}{log(S_i)}, \quad (15)$$

where $log(n) - log(\alpha T_i) > 0$. Thus, we can design a rejection sampling algorithm to generate from $p(\beta | \alpha, D)$ by using the prior $p(\beta)$ as the i*mportance function* as in Smith and Gelfand (1992). In other words, we can draw β from the prior and accept it with probability

$$\frac{\mathcal{L}(\beta;\alpha,D)}{\mathcal{L}(\,\widehat{\beta}\,;\alpha,D\,)}.$$

Once a sample is obtained from the joint posterior $p(\alpha,\beta|\,D)$, the posterior distribution $p(\lambda_u|\,D,\,S_i)$ and the variance $V(\lambda_u|\,D,\,S_i)$ can be computed from this sample. In finding the optimal design we need to obtain the preposterior variance

$$\mathrm{E}_D[\,V(\lambda_u|\,D,\,S_i)] = \int V(\lambda_u|\,D,S_i)\;p(D|S_i)\,dD,$$

which can be written as

$$\mathrm{E}_D[\,V(\lambda_u|\,D,\,S_i)] = \int\int V(\lambda_u|\,D,S_i)p(D|\lambda_u,S_i)\;p(\lambda_u)\,d\lambda_u\,dD. \tag{16}$$

We note that in the above $V(\lambda_u|\,D,S_i) = V(\lambda_u|\,T_i,S_i)$ is evaluated via the Gibbs sampler for a given design $d = S_i$. Since the integral in (16) is not available in closed form, we can use a Monte Carlo average to evaluate it. More specifically, for each generated value of $(\alpha_r,\ \beta_r)$, $r = 1,\ldots,R$, from the prior we can generate λ_i and T_i. Based on the each generated value of T_i, using the Gibbs sampler we can obtain the posterior variance $V[\lambda_u|T_i^r,\,S_i]$ and compute preposterior variance for S_i using the Monte Carlo average

$$\frac{1}{R}\sum_{r=1}^{R} V[\lambda_u|T_i^r,\,S_i]. \tag{17}$$

Note that optimal one-point design can be obtained by minimizing (17) with respect to S_i. This Monte-Carlo setup can modified for the general ALT design problem as discussed next.

4.1 Standard Monte Carlo Approach for Bayesian ALT Designs

Erkanli and Soyer (2000) consider a general fixed design for ALTs where m distinct stress levels are used and n_i items are tested at the stress level S_i such that $n = \sum_{i=1}^{m} n_i$ is a predetermined number. The design problem is then to select

(i) $m \le n$, the number of distinct stress levels;

(ii) the accelerated stress levels S_i, $i = 1,\ \ldots,m$; and

(iii) the number of items tested at each stress level, n_i, $i = 1, \ldots, m$

in such a way that expected utility is maximized. In the above, a specific design is given by $d = \{m,\ S_i,\ n_i, i = 1, \ldots, m\}$. As in Section 2.1, the authors consider the power law model where one is interested in making inference about λ_u, the failure rate at the use stress environment.

As before, we assume that there is no censoring in the ALT and we choose the design minmizing the preposterior variance of λ_u. In other words. the evaluation of the optimal design requires the computation of posterior variance $V(\lambda_u|\, D,\, d)$, where $D = \{T_1, \ldots T_m\}$ and $T_i = \sum_{j=1}^{n_i} x_j$ is the total time on test at stress environment S_i. As in the previous case evaluation of $V(\lambda_u|\, D,\, d)$ requires use of MCMC methods.

The following algorithm has been suggested by Erkanli and Soyer (2000) for standard Monte Carlo evaluation of the preposterior variance $\mathrm{E}_D[\, V(\lambda_u|\, D,\, d)]$:

Step 1: Choose $d = \{m,\ S_i,\ n_i, i = 1, \ldots, m\}$
Step 2: Generate $(\alpha_r,\ \beta_r)$ from the prior $p(\alpha,\ \beta)$, $r = 1, \ldots, R$
Step 3: For $i = 1, \ldots, m$, generate T_{ir} from $p(T_i|\lambda_i)$ using the power law
Step 4: For each $D_r = \{T_{1r}, \ldots, T_{mr}\}$ evaluate $V[\lambda_u|D_r,\ d]$ using MCMC.
Step 5: Compute preposterior variance for d using Monte Carlo average

$$\frac{1}{R}\sum_{r=1}^{R} V[\lambda_u|D_r,\ d] \tag{18}$$

Step 6: Go to step 1 and repeat the steps 2-5 for a different design d.

The optimal design is selected as the d with the minimum value of (18).

4.2 A Surface Fitting Algorithm for Bayesian ALT Designs

As noted by Erkanli and Soyer (2000), the implementation of the above Monte Carlo approach is not computationally efficient. The approach requires, for each level of the design variable, R draws from the statistical model and, for each draw, evaluation of the posterior variance using MCMC methods that typically requires large number of iterations. Thus, the approximation in (18) may require a large scale computational effort. Especially, this will be inefficient for the case of multiple stress designs where we need to specify m optimal stress environments. To avoid the potential computational burden

the authors suggested a curve fitting approach proposed by Muller and Parmigiani (1995) for finding optimal ALT designs. This approach facilitates preposterior analysis by replacing the expectation step with a smoothing step. The posterior variance is evaluated as $v_r = V(\lambda_{ur} | D_r, d_r)$ for each simulated experiment $\{d_r, \alpha_r, \beta_r, D_r\}$ and a smooth surface $L(d)$ is fitted to the points (d_r, v_r) and the optimal design is found by minimizing the fitted surface $L(d)$.

The proposed approach by Erkanli and Soyer (2000) for the ALT design problem with $d = \{m, S_i, n_i, i = 1, \ldots, m\}$ and $\boldsymbol{\theta} = (\alpha, \beta)$ is as follows :

Step 1: Select designs d_r, $r = 1, ..., R$.

Step 2: Draw R points $(D_r, \boldsymbol{\theta}_r)$ from the density $p(D, \boldsymbol{\theta} | d_r)$. This is done by independently generating $(T_{1r}, T_{2r}, \ldots, T_{mr})$ from $p(T_i | \lambda_i, d_r)$ for $i = 1, \ldots, m$.

Use MCMC to evaluate $v_r = V(\lambda_{ur} | D_r, d_r)$ and record sample points (d_r, D_r, v_r).

Step 3: Fit a surface $L(d)$ to the points (d_r, v_r).

Step 4: Find the minimum over d of $L(d)$.

The above setup assumes that total number of n items will be tested in an ALT. It is important to note that d consists of both discrete and continuous components. Given n, the discrete components of d, $\{m, n_i, i = 1, \ldots, m\}$ are constrained as $n = \sum_{i=1}^{m} n_i$ and $m \leq n$. Erkanli and Soyer (2000) considered the case of $m = 2$ point designs which are shown to be optimal for relationships such as power law model with two parameters [see for example, Chaloner and Larntz (1992)].

For the case of two-point designs, an alternate design strategy is to consider a sequential design. The general $m-$ stage sequential design problem for ALT is shown in Figure 2. The surface fitting approach conceptually can be adopted to the sequential problem, but as the number of stages increasing the dimension of the surfaces that we fit at each stage also increases and this makes the implementation quite difficult. Thus, the proposed approach was extended in Erkanli and Soyer (2000) for only two stage design problems using a setup similar to what is considered in Erkanli, Soyer and Angold (1998) for prevalence estimation.

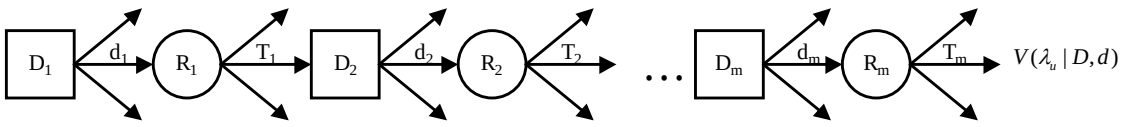

Figure 2. Decision Tree for the $m-$ stage Design Problem.

4.3 An Illustration of the Curve Fitting Method

In this section we will illustrate the implementation of the curve fitting algorithm introduced in the last section to a one-point design problem using exponential lifetimes and a power law model. In the illustration, we used the local regression model *Loess* of Chambers and Hastie (1992) in Step 3. Any one-dimensional smoothing method such as splines can be used for this purpose.

For illustrative purposes, we will assume independent gamma priors for α and β such that $\alpha \sim Gamma(20, 1000)$ and $\beta \sim Gamma(3, 1)$. We will assume that at the optimal stress level, $n = 2$ items will be tested and the design space consists of the stress range $S \in (1.05, 11)$ where $S_u = 1.05$ is the use-stress. As previously discussed, our goal is to find the optimal stress level S_i in such a way that the preposterior variance of λ_u is minimized. In what follows, as an alternative, we will be interested in finding the optimal stress level minimizing the preposterior variance of $\eta_u = log\lambda_u$ so that our results are comparable to those of Soyer and Vopatek (1995) where linear Bayesian designs are obtained.

We note that under the specified form of priors, we still need to use the Gibbs-rejection algorithm to evaluate the posterior variance $V(\eta_u | T_{ir}, d_r)$ in Step 2 of the curve fitting algorithm for each design point $d_r = S_i,\ r = 1, \ldots, R$. To find the optimal stress level we choose $R = 1000$ design points in the range $(1.05, 11)$ in Step 1 of the algorithm. Next 1000 random vectors $(\alpha_r,\ \beta_r,\ T_{ir}\,\}$ are simulated using the joint distribution $p(\alpha,\ \beta)\, p(T_{ir} | \alpha,\ \beta)$, where $(T_{ir} | \alpha,\ \beta) \sim Gamma(n,\ \lambda_{ir})$ and $\lambda_{ir} = \alpha S_i^{\beta}$. After implementing the Gibbs sampler for each generated data point and evaluating the posterior variance $V(\eta_u | T_{ir}, d_r)$ for $r = 1, \ldots 1000$, we can fit a nonparametric regression curve to the points $\Big(d_r = S_i, v_r = V(\eta_u | T_{ir}, d_r)\Big)$. As discussed before, this is equivalent to taking the expectation of the posterior variance with respect to T_{ir}. In Figure 3 we present the nonparametric approximation to preposterior variance of η_u, that is, $V(\eta_u | S_i)$ As can be seen from the figure the minimum of the curve is around 2.4-2.5. We note the flatness of the fitted curve around the minimum. This is due the fact that the loss function does not consider any costs associated with testing. Also, we note the variation in the preposterior variance which is due to the large variance of β. These results are very similar to the findings of Vopatek (1992) which are based on linear Bayesian methods. The results are also pretty robust to the choice of R as long as $R > 500$.

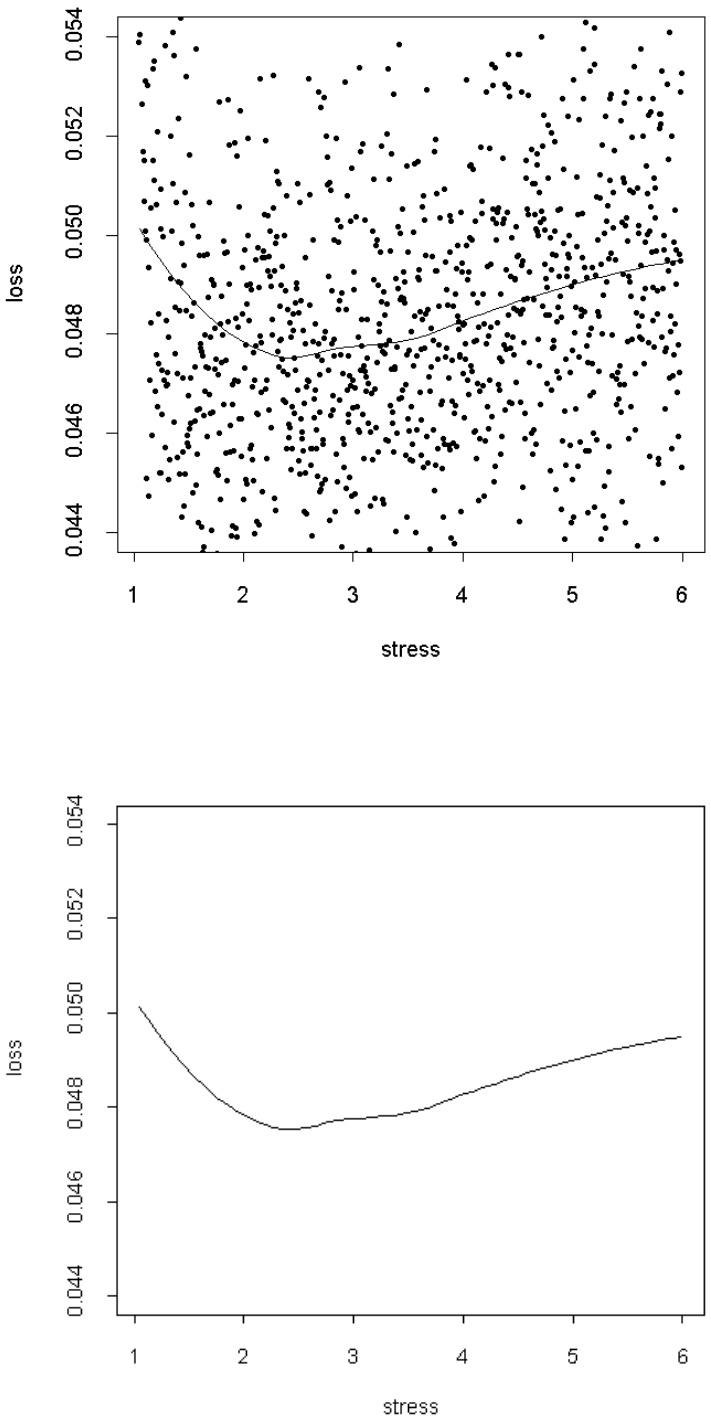

Figure 3. Nonparametric Approximation to $V(\eta_u|\, S_i)$ with $n = 2$.

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